Arithmetic progression with greates common difference.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

For given natural $n \geq 2,$ among increasing arithmetic progression $x_1, x_2, \ldots, x_n$ such that $x_1^2 + x_2^2 + \ldots + x_n^2 = 1,$ find arithmetic progression with greates common difference $d.$

Solution.

Let $x_1, x_2, \ldots, x_n$ be arbitrary increasing arithmetic progression $x_1, x_2, \ldots, x_n$ such that $x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$ Since $x_k = x_1 + (k-1)d, k = 1, 2, \ldots, n$ then $x_1^2 + x_2^2 + \ldots + x_n^2 = 1 \Leftrightarrow$

$x_1^2 + (x_1 + d)^2 + (x_1 + 2d)^2 + \ldots + (x_1 + (n-1)d)^2 = 1 \Leftrightarrow$

$nx_1^2 + 2x_1 d(1 + 2 + \ldots + n - 1) + d^2\left(1^2 + 2^2 + \ldots + (n-1)^2\right) = 1 \Leftrightarrow$

$nx_1^2 + x_1 d(n-1)n + d^2 \dfrac{(n-1)n(2n-1)}{6} = 1 \Leftrightarrow x_1^2 + x_1 d(n-1) = \dfrac{1}{n} - d^2 \dfrac{(n-1)(2n-1)}{6} \Leftrightarrow$

$\left(x_1 + \dfrac{d(n-1)}{2}\right)^2 = \dfrac{1}{n} - \dfrac{d^2(n^2-1)}{12}$ implies $d^2 \leq \dfrac{12}{n(n^2-1)} \Leftrightarrow$

$d \leq \dfrac{2\sqrt{3}}{\sqrt{n(n^2-1)}}$ (since $d > 0$). For $d = d_* = \dfrac{2\sqrt{3}}{\sqrt{n(n^2-1)}}$ we obtain

$\left(x_1 + \dfrac{d_*(n-1)}{2}\right)^2 = 0 \Leftrightarrow x_1 = -\dfrac{d_*(n-1)}{2} = -2\sqrt{3}\sqrt{\dfrac{n-1}{n(n+1)}}.$

So, arithmetic progression $x_k = -2\sqrt{3}\sqrt{\dfrac{n-1}{n(n+1)}} + \dfrac{2\sqrt{3}\,(k-1)}{\sqrt{n(n^2-1)}}, k = 1, 2, \ldots, n$

satisfy $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$ and maximize common difference $d,$ i.e. $\max d = \dfrac{2\sqrt{3}}{\sqrt{n(n^2-1)}}.$